WELCOME TO THE WORLD OF MATHEMATICS -- WHERE ANYTHING IS POSSIBLE!

¡BIENVENIDO AL MUNDO DE LAS MATEMÁTICAS, DONDE TODO ES POSIBLE!

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In my Keynote Address to PME-NA 45, I offer an embodied framework for naming what makes mathematics powerful for mathematicians and scientists, yet intractable for many learners. The essential claim is this: Students reside in the Real World, where math is grounded, embodied and meaningful, while mathematics resides in the ungrounded, disembodied realm of the UnReal World. To make all educational experiences meaningful, I consider ways to prepare students to be tourists to the UnReal World, such as progressive formalization and immersion in eXtended Reality (XR). Even so, educators must remember that learners remain citizens of the Real World even when visiting the UnReal World. I share examples of how embodied learners make sense of UnReal things, method of making bridges between these worlds, and concerns that entrenched assessment practices neglect the nonverbal ways of knowing expressed by embodied learners.

Welcome to the world of mathematics -- where anything is possible! This is both wonderful (for mathematicians) and terrible (for students). *Wonderful*, in that mathematics delivers a set of formal systems with precise and powerful rules for generating (and verifying) mathematically correct statements to model patterns, both real and imagined *Terrible*, because the truths that are generated are detached from the Real World and frequently defy the intuitions and expectations students have developed from their lived experiences. I offer this frame because my scholarship is positioned with a guiding value: **All educational experiences should be *meaningful* to students.** I arrive at this based on ethical considerations that educational institutions must be committed to improving students' lives and futures, and based cognitive considerations that meaningful information is more readily accessed, retained, and applied (Bransford et al., 2000). I refer to this focus on meaningful learning as **achieving a grounded understanding** (Nathan, 2014). Ideas and symbols that are abstract and unfamiliar become grounded for learners when they are connected to one's lived and felt experiences, including those ideas and symbols that are embodied through one's perceptions, actions, and social and cultural practices.

1. Students Reside in the Real World; Mathematics in the UnReal World

Mathematics education encounters an apparent enigma: If people naturally do math as they go about their lives, what makes learning math so difficult? The quandary falls away when we recognize that the math people do is what I call **Grounded and Embodied Math**, or **GEM**. It is the math that people *can* do because as embodied beings, people are grounded in their lived, contextually bound experiences. This is the mathematics of the **Real World (RW)** where people -- myself included! -- actually reside. Here in the RW, things have mass, and extent; operations take time and energy; capacities are limited, space confined. Hardly anything (perhaps not anything) is truly linear, exact, discrete, or known with certainty. In the RW where GEM occurs, all things are *not* possible. Herein lies a central challenge for math education.

One can also imagine a different world, **the UnReal World (unRW)**, where **UnGrounded and disEmbodied Math (unGEM)** is relevant and has meaning. This is a realm filled with

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dimensionless points, lines of infinite length, perfect correlations, and certainty.

*I use these terms, **RW, unRW, GEM, and unGEM** -- cumbersome as they may be -- to help firm up my thoughts about the current challenges facing math education and some of the promising pathways forward.*

When we teach math, we are really asking *two* things of students:

I. Master the formal systems of notation that describes the behavior of objects that adhere to the rules of unGEM; and

II. Become a resident of unRW, so that unGEM becomes meaningful.

Our educational institutions predominantly support students only with the first of these, and largely ignores the second. I believe it is imperative that we consider both propositions because there are many documented occasions where students can reliably apply the rules of unGEM (consistent with Proposition I), even as the formal statements of unGEM remain meaningless with respect to the world in which they reside (counter to Proposition II; e.g., Koedinger & Nathan, 2004; Landy & Goldstone, 2007). More commonly, students adhere to Proposition I by mastering unGEM even when they can demonstrate their understanding of the mathematical ideas to which the unGEM symbols refer. This is observed, for example, when students who speak different languages exhibit nonverbal ways of effectively expressing their mathematical reasoning during collaboration, thus circumventing unfamiliar terms (Swart et al., 2021).

2. Preparing Tourists to the UnReal World

One reason for this disconnect is a lack of *common ground* between phenomena in the RW and the unRW (Alibali et al., 2019). Naming the dichotomy between the grounded and embodied nature of the RW and the ungrounded and disembodied nature of the unRW helped me frame my goals in terms of an educational question: *How can we support students to become competent tourists in unRW*? I am aware of three promising ways (which need not be mutually exclusive):

1. *Progressive formalization*. This builds on the natural ability of humans to incrementally ground from concrete experiences to idealized descriptions (i.e., abstractions) of those experiences by establishing a new grounding of previously unfamiliar ideas in successive iterations. For example, manipulating physical objects (i.e., performing operations on RW objects) often provides the necessary grounding when first encountering numbers, while numbers later ground arithmetic operations, arithmetic comes to ground algebraic expressions, and so on. Realistic Mathematics Education (Gravemeijer & Doorman, 1999; Van den Heuvel-Panhuizen & Drijvers, 2020) and Concreteness Fading (Bruner, 1966; Fyfe et al., 2014; Ottmar & Landy, 2017) are two approaches to progressive formalization in math education with substantial empirical support. These approaches incrementally shifts RW experiences towards unGEM of unRW by successively stripping away the perceptual richness of RW actions and objects, so they come to serve as increasingly idealized representations of the original experience.

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2. *Professional Vision*. Changes in one's perceptions and language through instruction enables one to see and describe RW phenomena through a lens of unGEM (Goodwin, 1994; Stevens & Hall, 1998). Powerful examples of this are seen in Bob Moses' Algebra Project (Moses & Cobb, 2002), Learning Along Lines (Ma, 2016; Taylor, 2017), and Dallas Math Walks (Wang & Walkington, 2023). These approaches brings unGEM into RW by grounding unRW concepts in students' lived perceived and embodied experiences.

3. *Immersion in Microworlds*. This invites learners into spaces where objects only behave according to rules of unGEM. The includes approaches with varying degrees of immersion, from tablets that present manipulable mathematical objects for number (TouchMath; Sinclair & Heyd-Metzuyanim, 2014), algebraic expressions (Graspable Math; Ottmar et al., 2015), and geometric objects (Carbri, GeoGebra), to eXtended Reality (XR) environments (including VR and AR) that sensorially immerse one in unRW spaces (Dimmel et al., 2023; Johnson-Glenberg & Megowan-Romanowicz, 2017; Walkington et al., 2022). These approaches puts residents of RW in unRW to adopt the practices of unGEM; and, ideally, reflect upon those practices when they return to RW.

Each of these approaches rests on the amazing plasticity of the human neural system, and a realization that with the appropriate sensorimotor experiences, we are able to (temporarily) realign our expectations for how things ought to behave in the world. Examples such as the rubber hand illusion (National Geographic, 2015), illustrate the range of situations to which humans are able to adjust.

3. We Remain Citizens of the Real World Even When we Visit the UnReal World

As educators and educational researchers, it is important to keep in mind that people are inclined to apply the rules and expectations that have successfully guided their RW experiences even when visiting the unRW. To illustrate, humans -- mathematicians included, LOL! -- are inclined to interpret phenomena of the unRW through a GEM lens. Thus, numbers *are* locations in space, operations *are* physical actions on mathematical "objects," limits *are* fictive motion (Lakoff & Núñez, 2000), logic *is* a sequence of cause and effect, and set theoretic statements *are* idealizations of containers for collecting, categorizing and classifying objects (Johnson, 1980). In visual art, M. C. Escher's *Ascending and Descending,* intrigues us because it offers a twist (literally and actually) on our experiences of gravity the RW. Even the *Theory of Relativity,* arguably the paradigmatic case of thinking "outside the box," is said to have its origin in the mundane nature of Einstein's contemplations of the variations in train schedules across Europe as he commuted to and from his job at the patent office in Bern (Galison, 2004). It is not that people are unable to perceive unGEM objects, but that they will project their RW interpretations onto objects from unGEM. We see this abound from work analyzing students' math errors (e.g., Landy & Goldstone, 2007; Koedinger & Nathan, 2004; VanLehn, 1990) demonstrating that these errors often arise when people try to make unGEM objects conform to *their* RW. Thus, it is important to acknowledge -- and accommodate -- learner's natural tendencies.

This should not be taken to suggest that people are basically literal thinkers, or lack imagination. Contrarily, people exhibit great imagination in their contributions to science, technological innovation, and the arts. Rather, these realizations highlights how people are naturally inclined to use their experiences to rationalize problem spaces, and, further, to point out

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that even people's imagination is not limitless; it is tethered to one's RW, embodied experiences. Artists such as Escher (1972; see note 1) in the visual realm, and Pink Floyd in music (Waters, Wright, Mason, & Gilmour, 1970; see note 2) demonstrate that even when people imagine behaviors that violate the laws of the RW, they do so while standing on *terra firma*.

While we all live in the RW, there are actual residents of the unRW: ChatGPT*n* (currently at version 4) is one example, and so is your smartphone. These residents are comfortable with living in ungrounded, formal spaces. Borges' (1962/ 2007) *Library of Babel* is one such space. Borges imagined an institution that contains every conceivable book by generating every orthographic variations of the alphabet, regardless of whether it is readable or accurate (e.g., the entire contents of my future talk). Personally, as a trained mathematician, I like to visit unRW and partake in unGEM, but as a trained engineer, I do not wish to reside there permanently.

One of our superpowers as human beings is our ability to process *all* events from a grounded and embodied perspective. Just because humans have developed strings of symbols to represent systems of language, culture, art, and mathematics, does not mean that we take comfort in their arbitrary associations to ideas and events. Unlike AI, we do not simply store these formalisms verbatim or process them blindly. Rather, people project meaning onto these symbol systems because of their cultural associations, embodied affordances, and the mental simulations they invoke (Bransford et al., 1972; Gallese & Sinigaglia, 2011; Glenberg & Robertson 2000). This tethering to worldly experiences is one of the advantages people have over AI systems and it is one reason why humans excel in expressing themselves through art and mathematics. Practice and feedback from the RW affords humans opportunities to get better at accurately recognizing and generating images that represent RW objects. Ironically, generative AIs trained on synthetic data produced by other generative AIs rapidly degrade with practice and feedback, and the images they produce become further removed from real-looking objects (Alemohammad et al., 2023). unGEM, operating in its own self-referential world, becomes self-consuming and increasingly absurd (Eisenstein, 2023).

4. Final Thoughts: Assessing the Understanding of Grounded and Embodied Learners

Since its inception, the PME community has acknowledged a key insight by focusing on the *Psychology* of Mathematics Education. Educational practitioners and leaders, working with members of the education research community, must bridge GEM and unGEM. At its core, these bridging activities necessitate a continued understanding of the psychological needs of students and teachers to adopt the reasoning and practices of an unReal World that has tremendous utility for modeling the RW and gifting it with innovations that further the Public Good. Existing and emerging educational practices and technologies offer promising inroads, but ultimately depend upon an understanding of human thinking, human development and human behavior.

This is made imminently clear by examining a persistent misalignment between our emerging conceptualizations of student knowledge and current knowledge assessment practices. Once we recognize the grounded and embodied nature of people's mathematical thinking and learning, including students' nonverbal and nonsymbolic ways of knowing and expressing their thinking, it is clear that formative and summative knowledge assessment practices must follow suit. Fortunately, teachers who themselves engage in and reflect upon their own embodied mathematical behaviors seem to become more inclined to notice students' nonverbal ways of expressing their mathematical reasoning and to address students' embodied behaviors in their formative assessment practices (Sung, Swart & Nathan, 2021). Considerations of both formative and summative assessment practices designed with grounded and embodied learners in mind

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offer some of the most promising next steps for advancing mathematics education for all learners.

Author Notes

Note 1. M. C. Escher's lithograph *Waterfall* (Dutch: *Waterval*; Escher, 1972) gives the appearance of a normal flow of water that endlessly cycles upward or downward. The watercourse uses two Penrose triangles to create the illusion. The Penrose triangle was designed by Oscar Reutersvärd in 1934, and independently discovered by Roger Penrose in 1958 (Penrose & Penrose, 1958).

Note 2. Pink Floyd's (1970) "Echoes" is a composition on the *Meddle* album that runs over 23 minutes. The composers used the Shepard-Risset Glissando, a variant of the Shepard Tone created by psychologist Roger Shepard (1964), which can be heard at the end of composition, beginning just after the 22-minute mark in the Meddle version, https://www.youtube.com/watch?v=KBca3xf-j3o.

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